

Thermodynamics of phase transitions in AdS black holes

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Abstract

Based on fundamental concepts of thermodynamics we examine phase transitions in black holes defined in Anti-de Sitter (AdS) spaces. The examples of Reissner-Nordstrom (RN) AdS and Kerr AdS black holes are discussed. Considering these black holes as grand-canonical ensembles we exhibit and classify a phase transition defined by the discontinuity of specific heat as well as volume expansion coefficient and compressibility. Using standard Ehrenfest's scheme we prove that such phase transitions between the 'smaller' and 'larger' mass black holes are second order. We also provide the explicit expressions for the critical mass and critical temperature. Also, the distinction between this phase transition with the well known Hawking-Page transition is highlighted.

1 Introduction

It is well known that black holes behave as thermodynamic systems. The laws of black hole mechanics become similar to the usual laws of thermodynamics after appropriate identifications between the black hole parameters and the thermodynamical variables [1]. The thermodynamic properties of black holes were further elaborated when Hawking and Page discovered a phase transition in the Schwarzschild black hole defined in Anti-de-Sitter (AdS) space [2]. Since then, this subject has been intensely researched for the last few decades [3]-[15]. In spite of this activity, however, several issues remain unanswered. One of the reasons for this is that the original approach [2], based on a path integral formulation, was never connected to the conventional (Gibbsian) way of discussing phase transitions in thermodynamic systems.

The Gibbsian approach to the phase transitions is the standard approach based on the Clausius-Clapeyron-Ehrenfest's equations [16, 17]. These equations allow for a classification of phase transitions as first order or continuous (higher order) transitions. Indeed this specific issue concerning the nature of phase transition has never been systematically addressed in this framework. Another intriguing point is the possibility of generalizing the results of [2] to other black holes. Although there have been some attempts [12, 13] no quantitative results are available that, in appropriate limits, reproduce the corresponding expressions given in [2]. Consequently the claimed generalizations and/or extensions of [2] remain highly debatable and worthy of further investigations.

In this paper we address these issues. We develop an intuitively simple approach based on standard thermodynamics which to some extent parallels the path integral description of Hawking and Page [2]. As a nontrivial exercise we go beyond the Schwarzschild Anti-de-Sitter black hole and perform the analysis for both charged and rotating (i.e. Reissner-Nordstrom

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(RN) AdS and Kerr AdS) black holes. Some key results obtained in [2] are generalized for these two cases. In appropriate limits, they reproduce the qualitative as well as quantitative results of the Schwarzschild AdS example. Then we perform a thorough Gibbsian analysis to study phase transition in AdS black holes. This phase transition is characterized by the divergence of the specific heat at the critical temperature. The sign of specific heat is different in two phases and they essentially separate two branches of AdS black holes with different mass/ horizon radius as mentioned in [2]. The branch with lower mass (horizon radius) has a negative specific heat and thus falls in an unstable phase. The other branch with larger mass (horizon radius) is locally stable since it is associated with the positive specific heat and also positive Gibbs energy. We show such transitions are second order. In order to do that we analytically check the validity of Ehrenfest's like equations for black holes [18]-[20] infinitesimally close to the critical point. Since all the relevant quantities (i.e. specific heat, volume expansion coefficient and compressibility) diverge at the phase transition point we develop special techniques to study these equations in this diverging region.

As a point of caution we would like to mention that the phase transition studied in this paper should not be confused with the well known Hawking-Page phase transition. The latter is a transition between the AdS space and the globally stable black hole phase. This has nothing to do with the discontinuity of specific heat of AdS black holes which is considered in this paper. In fact the Hawking-Page phase transition takes place where the Gibbs energy changes its sign whereas we are studying the case where Gibbs energy is maximum. The phase transition considered in this paper is a transition from the lower mass (unstable) to a higher mass (locally stable) black hole phase. This occurs at a temperature which is less than the Hawking-Page phase transition temperature.

2 Gibbsian approach to phase transition

Hawking and Page first investigated the thermodynamic properties of Schwarzschild AdS black hole [2]. Let us briefly summarize their findings. They found that below a certain minimum temperature

$$T_0 = \frac{1}{2\pi} \sqrt{\frac{3}{l^2}} \quad (1)$$

the thermal radiation in AdS space is stable and does not collapse to form a black hole. Here l is related to the cosmological constant Λ by $\Lambda = -\frac{3}{l^2}$. However, at $T > T_0$, depending upon the mass, a newly formed black hole can be either stable or unstable. This critical mass is given by

$$M_0 = \frac{2l}{3\sqrt{3}} \quad (2)$$

The smaller mass ($M < M_0$) black hole falls in the unstable region since it has a negative specific heat. There exists another phase where the mass of this black hole is large ($M > M_0$). This is in a locally stable phase since the specific heat and free energy are positive. In fact a smaller mass unstable black hole may tend to switch over to the stable large black hole phase by absorbing radiation from the thermal AdS space. This is equivalent to minimizing the free energy for the smaller black hole. There also exists a temperature T_1 ,

$$T_1 = \frac{1}{\pi l} \quad (3)$$

such that for any T ($T_0 < T < T_1$) the free energy is always positive so this configuration would reduce its free energy if the black hole evaporated completely. In fact at $T = T_1$ Gibbs

free energy changes its sign. This transition is latter known as Hawking-Page phase transition between the thermal AdS space and globally stable black hole phase. A similar phase transition was also found by Witten as the confinement-de confinement transition of the Yang-Mills theory in the AdS-CFT correspondence [21, 22].

In this paper we are concerned with the phase transition occuring at temperature T_0 . It is well known that at this temperature specific heat of AdS black hole is discontinuous [8]. We further analyze the nature of this phase transition using Ehrenfest's scheme (this issue is addressed in Section-3).

Before we start giving our algebraic details, let us mention that throughout this letter, the various black hole parameters M, Q, J, S, T should be interpreted as $\frac{M}{l}, \frac{Q}{l}, \frac{J}{l}, \frac{S}{l^2}, Tl$ respectively, where the symbols have their standard meanings. Also, with this convention, the parameter l no longer appears in any of the equations [20].

To follow a Gibbsian approach it is customary to define the Gibbs free energy. For RN-AdS black hole this is defined as $G = M - TS - \Phi Q$ where the last term is the analog of PV term in conventional systems. The expressions for mass (M), temperature (T), charge (Q) and the entropy (S) are respectively given by [20]

$$M = \frac{\sqrt{S}[\pi(1 + \Phi^2) + S]}{2\pi^{\frac{3}{2}}} \quad (4)$$

$$T = \frac{\pi(1 - \Phi^2) + 3S}{4\pi^{3/2}\sqrt{S}}, \quad (5)$$

$$Q = \Phi\sqrt{\frac{S}{\pi}} \quad (6)$$

$$S = \pi r_+^2. \quad (7)$$

where Φ is the electric potential and r_+ is the radius of the outer event horizon. Using these expressions one can easily write G as a function of (S, Φ) ,

$$G = \frac{\sqrt{S}}{4\pi^{3/2}} (\pi(1 - \Phi^2) - S), \quad (8)$$

Considering a grand canonical ensemble (fixed Φ) it is now straightforward to find the specific heat at constant potential (C_Φ), which is the analog of C_P (specific heat at constant pressure) in conventional systems. This is found to be

$$C_\Phi = T \left(\frac{\partial S}{\partial T} \right)_\Phi = 2S \frac{\pi(1 - \Phi^2) + 3S}{3S - \pi(1 - \Phi^2)}, \quad (9)$$

With this machinery we are in a position to describe phase transitions in AdS black holes within the scope of standard thermodynamics. As a first step we plot Gibbs free energy, entropy and specific heat against temperature in figure 1. All these plots have a common feature; they exist only when the temperature of the system is greater than the critical temperature ($T \geq T_0$). At T_0 they all show nontrivial behavior and as we subsequently prove that there is a well defined phase transition at this temperature.

We first provide a qualitative agreement between the results found here and those obtained by the path integral formulation in [2]. This will be followed by explicitly reproducing and generalizing (1,2, 3).

First, consider the $G - T$ plot. It has two wings which are joined at temperature T_0 . At the upper wing, because of the positive definite values of G , this system falls in an unstable phase (Phase-1). At T_0 Gibbs free energy is maximum which implies that the system is most unstable at this point. Therefore it cannot stay at T_0 for long and eventually passes to the locally stable

phase (Phase-2) by minimizing its free energy. This is achieved by following the other (lower) wing. In addition to that at $T = T_0$ the entropy-temperature plot is continuous and also changes its direction. The slope of this plot is negative for entropy lower than a critical value (S_0) while it is positive for all values higher than S_0 . Entropy being proportional to the mass of a black hole, phase-1 in $S - T$ plot corresponds to the lower mass (unstable) black holes while phase-2 belongs to black holes with higher mass (locally stable). This behavior is further exemplified in the plot of specific heat with temperature. The negative slope in $S - T$ results in a negative heat capacity in phase-1. As the system approaches the critical point, C_Φ diverges. Exactly at T_0 it flips from negative infinity to positive infinity. Through this phase transition the system emerges into a locally stable phase (phase-2) where the specific heat becomes positive. For $T < T_0$ black holes do not exist and what remains is nothing but the thermal radiation in pure AdS space.

Furthermore from the $G - T$ plot we find that G changes its sign at temperature $T = T_1$. The free energy is thus always positive for $T_0 < T < T_1$. For $T > T_1$ the system is globally stable. At T_1 we have the well known Hawking-Page transition.

We now give algebraic details. As a first step, the minimum temperature T_0 is calculated for the RN AdS space. This is found from the temperature at which C_Φ diverges (see fig.1). From (9) this occurs at the critical entropy,

$$S_0 = \frac{\pi}{3}(1 - \Phi^2). \quad (10)$$

Substituting this value in (5) we find the critical temperature

$$T_0 = \frac{1}{2\pi} \sqrt{3(1 - \Phi^2)}. \quad (11)$$

This temperature is the exact analogue of (1) for the RN-AdS example. In the charge less limit ($\Phi = 0$), this result goes over to (1)¹.

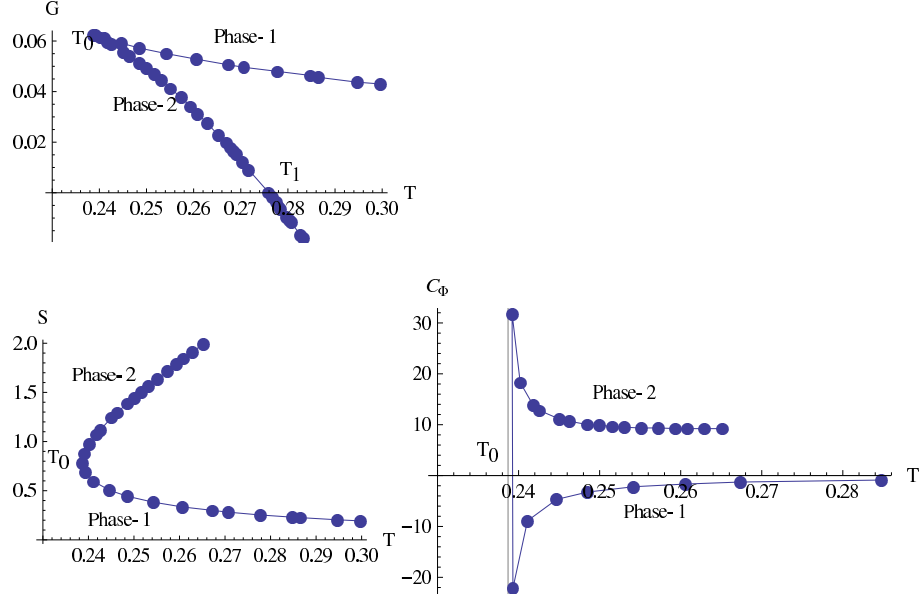


Figure 1: Gibbs free energy (G), specific heat (C_Φ) and entropy (S) plot for RN-AdS black hole with respect to temperature (T) for fixed $\Phi = 0.5$

Observe that (11) also follows by following the original analysis of [2]. Following this approach we first calculate the minimum temperature T_0 for the RN-AdS space-time. One can construct

¹Note that l does not appear since as explained before, it has been appropriately scaled out.

thermal states in RN-AdS space time by periodically identifying the imaginary time coordinate τ with period

$$\beta = \frac{4\pi^{3/2}\sqrt{S}}{\pi(1 - \Phi^2) + 3S} \quad (12)$$

which is the inverse Hawking temperature (T). For the maximum value of β (i.e. minimum value of T) at we set $[\partial\beta/\partial r_+]_{\Phi} = 0$. From this condition and by using (7) we find $S = \frac{\pi}{3}(1 - \Phi^2)$. Substituting this into (5) we find the corresponding minimum temperature for RN-AdS space time to be identical with (11).

The critical mass separating large and small black holes is now deduced. By substituting (10) in (4) we obtain,

$$M_0 = \frac{\sqrt{1 - \Phi^2}(2 + \Phi^2)}{3\sqrt{3}}. \quad (13)$$

In the charge less limit ($\Phi = 0$) the result (13) reproduces (2).

Finally, the value of T_1 is obtained from (8) and (5),

$$T_1 = \frac{\sqrt{1 - \Phi^2}}{\pi}. \quad (14)$$

Above this temperature as G is always negative, the RN-AdS black holes remain globally stable. If we make $\Phi = 0$, the temperature T_1 is identical with (3). Therefore we conclude that (11), (13) and (14) are the generalizations of (1), (2) and (3) for the RN-AdS case.

For the Kerr-AdS black hole one can easily perform a similar analysis and describe phase transition for that case. The graphical analysis shown above for RN-AdS black hole physically remains unchanged for the Kerr-AdS case. In particular one can calculate the inverse temperature

$$\beta^{Kerr} = \frac{4\pi^{\frac{3}{2}}[S(\pi + S)(\pi + S - S\Omega^2)]^{1/2}}{\pi^2 - 2\pi S(\Omega^2 - 2) - 3S^2(\Omega^2 - 1)} \quad (15)$$

and the condition $[\partial\beta^{Kerr}/\partial r_+]_{\Omega} = 0$ gives

$$(\pi + S)^3(3S - \pi) - 6S^2(\pi + S)^2\Omega^2 + S^3(4\pi + 3S)\Omega^4 = 0. \quad (16)$$

By substituting the positive solution of this polynomial in (15) one finds the minimum temperature T_0^{Kerr} . Furthermore for the Kerr-AdS black hole the heat capacity diverges exactly where (16) holds [19] which corresponds to the minimum temperature (T_0^{Kerr}). In the irrotational limit T_0^{Kerr} reproduces the known result (1). Results for M_0 and T_1 can also be calculated by a similar procedure. These expressions are rather lengthy and hence omitted. Moreover there are no new insights that have not already been discussed in the RN-AdS example.

In the remaining part of this paper we shall analyze and classify the phase transition at temperature T_0 by exploiting Ehrenfest's scheme. Since Hawking-Page transition at temperature T_1 is already studied extensively in the literature we do not include this part in our paper.

3 Nature of the phase transition

A look at the $(S - T)$ graph (fig.1) shows that entropy is a continuous at temperature T_0 . Consequently a first order transition is ruled out. However, the infinite discontinuity in specific heat (see fig.1) strongly suggests the onset of a higher order (continuous) phase transition. Under this circumstance, Ehrenfest's equations are expected to play a role. The derivation of these equations demands the continuity of specific entropy (S), specific charge (Q) and specific angular momentum (J) at the critical point. As a result these relations are truly local and only

valid infinitesimally close to the critical point. For a genuine second order phase transition both equations have to be satisfied [16, 17].

We start by considering the RN-AdS black hole. Ehrenfest's equations for this system are given by [20]

$$-\left(\frac{\partial\Phi}{\partial T}\right)_S = \frac{C_{\Phi_2} - C_{\Phi_1}}{TQ(\alpha_2 - \alpha_1)} \quad (17)$$

$$-\left(\frac{\partial\Phi}{\partial T}\right)_Q = \frac{\alpha_2 - \alpha_1}{k_{T_2} - k_{T_1}} \quad (18)$$

where, $\alpha = \frac{1}{Q} \left(\frac{\partial Q}{\partial T}\right)_\Phi$ is the analog of volume expansion coefficient and $k_T = \frac{1}{Q} \left(\frac{\partial Q}{\partial \Phi}\right)_T$ is the analog of isothermal compressibility. Their explicit expressions are given by [20]

$$Q\alpha = \frac{4\pi\Phi S}{3S - \pi(1 - \Phi^2)} \quad (19)$$

$$Qk_T = \sqrt{\frac{S}{\pi}} \frac{3\pi\Phi^2 - \pi + 3S}{3S - \pi(1 - \Phi^2)} \quad (20)$$

The L.H.S. of both Ehrenfest's equations (17) and (18) are found to be identical at the critical point S_0 . Using the defining relations (5,6) it is easily shown [20],

$$-\left[\left(\frac{\partial\Phi}{\partial T}\right)_S\right]_{S=S_0} = -\left[\left(\frac{\partial\Phi}{\partial T}\right)_Q\right]_{S=S_0} = \frac{2\sqrt{\pi}\sqrt{S_0}}{\Phi}. \quad (21)$$

In order to calculate the right hand sides, Φ must be treated as a constant (this is analogous to fixing pressure while performing an experiment). This would help us to re-express C_Φ (9), $Q\alpha$ (19), and Qk_T (20), which have functional forms $\frac{f(S)}{g(S)}$, $\frac{h(S)}{g(S)}$ and $\frac{k(S)}{g(S)}$ respectively, infinitesimally close to the critical point (S_0). Note that they all have the same denominator which satisfies the relation $g(S_0) = 3S_0 - \pi(1 - \Phi^2) = 0$. This observation is crucial in the ensuing analysis.

The expressions of C_Φ , $Q\alpha$ and Qk_T in the two phases ($i = 1, 2$) are respectively given by $C_\Phi|_{S_i} = C_{\Phi_i}$, $Q\alpha|_{S_i} = Q\alpha_i$ and $Qk_T|_{S_i} = Qk_{T_i}$. To obtain the R.H.S. of (17) we first simplify it's numerator:

$$C_{\Phi_2} - C_{\Phi_1} = \frac{f(S_2)}{g(S_2)} - \frac{f(S_1)}{g(S_1)} \quad (22)$$

Taking the points close to the critical point we may set $f(S_2) = f(S_1) = f(S_0)$ since $f(S)$ is well behaved. However since $g(S_0) = 0$ we do not set $g(S_2) = g(S_1) = g(S_0)$. Thus

$$C_{\Phi_2} - C_{\Phi_1} = f(S_0) \left(\frac{1}{g(S_2)} - \frac{1}{g(S_1)} \right). \quad (23)$$

Following this logic one derives,

$$\frac{C_{\Phi_2} - C_{\Phi_1}}{T_0 Q(\alpha_2 - \alpha_1)} = \frac{f(S_0)}{T_0 h(S_0)} = \frac{2\sqrt{\pi}\sqrt{S_0}}{\Phi} \quad (24)$$

and, similarly,

$$\frac{Q(\alpha_2 - \alpha_1)}{Q(k_{T_2} - k_{T_1})} = \frac{h(S_0)}{k(S_0)} = \frac{2\sqrt{\pi}\sqrt{S_0}}{\Phi}. \quad (25)$$

Remarkably we find that the divergence in C_Φ is canceled with that of α in the first equation and the same is true for the case of α and k_T in the second equation. From (21,24,25) the validity

of the Ehrenfest's equations is established. Hence this phase transition in RN-AdS black hole is a genuine second order transition.

Ehrenfest's set of equations for Kerr-AdS black hole are given by [18, 19]

$$-\left(\frac{\partial\Omega}{\partial T}\right)_S = \frac{C_{\Omega_2} - C_{\Omega_1}}{TJ(\alpha_2 - \alpha_1)}, \quad (26)$$

$$-\left(\frac{\partial\Omega}{\partial T}\right)_J = \frac{\alpha_2 - \alpha_1}{k_{T_2} - k_{T_1}}. \quad (27)$$

The expressions for specific heat (C_Ω), analog of the volume expansion coefficient (α) and compressibility (k_T) are all provided in [19]. Once again, they all have the same denominator (like the corresponding case for RN-AdS). Considering the explicit expressions given in [19] and using the same techniques we find that both sides of (26) and (27) lead to an identical result, given by,

$$l.h.s = r.h.s = \frac{4\pi^{\frac{3}{2}}(\pi + S_0 - S_0\Omega^2)^{\frac{3}{2}}(\pi + S_0)^{\frac{1}{2}}}{\sqrt{S_0}\Omega[3(\pi + S_0)^2 - S_0\Omega^2(2\pi + 3S_0)]}.$$

This shows that the phase transition for Kerr-AdS black hole is also second order.

4 Conclusions

We have established a formulation to discuss phase transition in AdS black holes following a conventional thermodynamical approach. This further clarified the interpretation of black holes as thermodynamic objects. Within the Gibbsian approach, some key results of Hawking and Page [2] were reproduced for the Schwarzschild AdS black hole, both at the qualitative and quantitative levels.

The analysis was generalized to other black holes. Specifically, considering a grand canonical ensemble, the critical temperature and mass of the charged (Reissner-Nordstrom) AdS black holes undergoing a phase transition were calculated. The phase transition discussed in this paper was defined by the discontinuity of specific heat, volume expansion coefficient and compressibility. The two phases were identified with black holes having 'smaller' and 'larger' mass/ horizon radius. The branch with smaller mass has negative specific heat and positive free energy, while, the other (larger mass) branch has positive specific heat and positive free energy (less than the free energy of smaller mass black holes). Thus it was a transition between an unstable phase to a locally stable phase. We also resolved the vexing issue of the nature of this phase transition. Although various relevant thermodynamic variables diverged at the critical point, we were able to devise a way of checking the Ehrenfest's relations appropriate for a second order transition. The exact validity of these relations confirmed the second order nature of the phase transition. Similar results for the rotating (Kerr) AdS black holes were outlined. Various features of the usual phase transition in the Schwarzschild AdS case were all found in the charged and rotating examples. The distinction of this phase transition with the standard Hawking-Page transition was also highlighted.

Our analysis strongly suggests that black holes are indeed governed by the concepts of thermodynamics. We believe that the present study would encourage to build a proper microscopic description of black hole phase phase transitions which is still not known.

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